THE MEAN VOLUME OF BOXES AND CYLINDERS CIRCUMSCRIBED ABOUT A CONVEX BODY

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ABSTRACT

The mean volume of boxes circumscribed about a convex body K of given volume is a minimum when K is a ball. This follows from a more general inequality, where the volume of circumscribed boxes is replaced by the product of quermassintegrals of the projections of K on appropriate lower dimensional subspaces.

Introduction

Let K be a convex body in Euclidean n-space E_n, $n \geqq 2$, that is, K is a compact, convex subset of E_n having nonempty interior. Let $g \in SO(n)$ be a proper motion, and let $u_1, \cdots, u_n$ be the orthonormal frame that is the image under g of the standard orthonormal basis of E_n. Let $n_1, \cdots, n_k$ be a partition of n, with $n = n_1 + \cdots + n_k$, where the n_i are integers satisfying $1 \leqq n_i \leqq n-1$, $i = 1, \cdots, k$. In what follows $E[n_1]$ will denote the n_1-dimensional subspace spanned by the first n_1 members of $u_1, \cdots, u_n$, $E[n_2]$ will denote the n_2-dimensional subspace spanned by the next n_2 members, and so forth. In other words, $E[n_{\alpha+1}]$ will denote the subspace spanned by $u_{n_1+\cdots+n_\alpha+1}, \cdots, u_{n_1+\cdots+n_{\alpha+1}}$, for $0 \leqq \alpha \leqq k-1$. Further, let $p_1, \cdots, p_k$ be integers satisfying $0 \leqq p_i \leqq n_i$, $i = 1, \cdots, k$. Associated with the orthogonal projection $K \mid E[n_\alpha]$ of K into $E[n_\alpha]$ is the p_α-th quermassintegral computed relative to that subspace and denoted by $\tilde{W}_{p_\alpha}(K \mid E[n_\alpha])$. The definitions and important properties of these quermassintegrals are found in [2]. For example, $\tilde{W}_0(K \mid E[n_\alpha])$ is the n_α-dimensional volume of $K \mid E[n_\alpha]$. Since we shall be dealing with products of these quermassintegrals it will be convenient to introduce the notation $\tilde{W}(K; n_1, \cdots, n_k; p_1, \cdots, p_k)$ for the product $\tilde{W}_{p_1}(K \mid E[n_1]) \cdots \tilde{W}_{p_k}(K \mid E[n_k])$. Letting dg represent the density for normalized Haar measure on $SO(n)$, the mean value of the product $\tilde{W}$ is given by

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$$(1) \qquad J(K; n_1, \cdots, n_k; p_1, \cdots, p_k) = \int \tilde{W}(K; n_1, \cdots, n_k; p_1, \cdots, p_k) dg,$$

where the integration is over all of $SO(n)$.

Observe that if $n_1 = n_2 = \cdots = n_k = 1$ and $p_1 = p_2 = \cdots = p_k = 0$, then $\tilde{W}(K; 1, \cdots, 1; 0, \cdots, 0)$ is the volume of the box circumscribed about K and having $\pm u_1, \cdots, \pm u_n$ as outward normals of its facets. Hence $J(K; 1, \cdots, 1; 0, \cdots, 0)$ is the mean volume of boxes circumscribed about K. As another case, suppose $n_1 = 1$, $n_2 = n - 1$ and $p_1 = p_2 = 0$. Then $\tilde{W}(K; 1, n-1; 0, 0)$ is the volume of the cylinder circumscribed about K with generators parallel to u_1 and base parallel to the hyperplane spanned by $u_2, \cdots, u_n$. Thus $J(K; 1, n-1; 0, 0)$ is the mean volume of right cylinders circumscribed about K.

Let $W_p(K)$ be the p-th quermassintegral of K itself. For example, $W_0(K)$ is the volume of K, and $nW_1(K)$ is the surface area of K. $W_n(K) = \omega_n$ is the volume of the n-dimensional unit ball. Our main result, giving an inequality between J and W_p, is the following:

THEOREM 1. *Let K be a convex body in E_n, and $n_1, \cdots, n_k$, $p_1, \cdots, p_k$ as above. If $p = p_1 + \cdots + p_k$, then we have*

$$(2) \qquad J(K; n_1, \cdots, n_k; p_1, \cdots, p_k) \geqq \frac{\omega_{n_1} \cdots \omega_{n_k}}{\omega_n} W_p(K).$$

If $p_i \neq n_i$ for at least two different values of i, then equality can hold in (2) only if K is a ball. Otherwise equality holds in (2) for all K.

Some special cases of the theorem are of interest. As pointed out above, $J(K; 1, \cdots, 1; 0, \cdots, 0)$ is the mean volume of boxes circumscribed about K. Using the fact that $\omega_1 = 2$ and $W_0(K) = V(K) =$ the volume of K, we obtain from (2)

$$(3) \qquad J(K; 1, \cdots, 1; 0, \cdots, 0) \geqq \frac{2^n}{\omega_n} V(K).$$

Equality can hold in (3) only if K is a ball. This generalizes a result of Radziszewski [4], obtained for the case $n = 2$, and later rediscovered by Chernoff [1]. A bit more generally, let $J_p(K)$ denote $J(K; n_1, \cdots, n_k; p_1, \cdots, p_k)$ with $n_1 = n_2 = \cdots = n_k = 1$, $p_1 = p_2 = \cdots = p_{n-p} = 0$, and the remaining p_α all equal to 1. Then (2) yields

$$(4) \qquad J_p(K) \geqq \frac{2^n}{\omega_n} W_p(K).$$

From the expression for the quermassintegrals of a box [2, p. 216], and using the invariance of dg, it is not difficult to see that (4) is an inequality relating the p-th quermassintegral of K with the mean value of the p-th quermassintegrals of its circumscribed boxes. In particular, $J_1(K)$ is proportional to the mean surface area of boxes circumscribed about K, so (4) gives an inequality between this mean and the surface area of K. This inequality was also established by Schneider [5], using spherical harmonics.

As another special case of Theorem 1, let $n_1 = 1$, $n_2 = n - 1$ and $p_1 = 0$, $p_2 = p \leqq n - 1$. Then $\tilde{W}(K; 1, n-1; 0, p)$ is the product of the altitude of a right cylinder circumscribed about K with the p-th quermassintegral of its base. In this case Theorem 1 gives an inequality between the mean of this product and $W_p(K)$. For example, with $p = 0$ we obtain

$$J(K; 1, n-1; 0, 0) \geqq \frac{2\omega_{n-1}}{\omega_n} V(K), \tag{5}$$

with equality holding only if K is a ball. The left hand side is the mean volume of right cylinders circumscribed about K. A choice of $p = 1$ leads to an inequality between the surface area of K and the mean lateral surface areas of circumscribed cylinders. These results for circumscribed cylinders were established by Knothe [3] for the case $n = 3$.

Our proof of Theorem 1 is by induction on n and p. Using a method similar to that in [3], we show that the validity of (2) with $p = 0$ follows from its validity for $p > 0$. Then by applying projection formulas of integral geometry, we show that (2) can be proved for $n = m$ and $p > 0$, provided it is known to be true for all $n < m$.

2. Preliminary lemmas

In proving the following preparatory lemmas, we see that the functionals $J(K; \cdots)$ bear a strong resemblance to the quermassintegrals in their behavior under projection and in forming parallel bodies.

LEMMA 1. *Let $n \geqq 2$ be fixed, and suppose* (2) *holds for $p > 0$. Then it follows that* (2) *is true with $p = 0$.*

PROOF. Let B be the unit ball, let $h \geqq 0$, and let $K + hB$ be the outer parallel set of K at distance h, that is, the vector sum of K and hB. Using the fact that vector addition commutes with projection, applying Steiner's formula for volume of the outer parallel set [2, p. 214], and dropping terms containing higher powers of h, we obtain

$$\tilde{W}_0(K + hB \mid E[n_\alpha]) \geqq \tilde{W}_0(K \mid E[n_\alpha]) + hn_\alpha \tilde{W}_1(K \mid E[n_\alpha]). \tag{6}$$

Now substitute successively $\alpha = 1, 2, \cdots, k$ in (6), multiply the resulting inequalities, and discard terms involving powers of h higher than one on the right hand side. Then integrate both sides of the resulting inequality over $SO(n)$. Using for the moment the more convenient notation $J(K; p_1, \cdots, p_k) = J(K; n_1, \cdots, n_k; p_1, \cdots, p_k)$, we obtain

$$\begin{aligned} J(K + hB; 0, \cdots, 0) \geqq J(K; 0, \cdots, 0) &+ hn_1 J(K; 1, 0, \cdots, 0) + \\ &+ hn_2 J(K; 0, 1, 0, \cdots, 0) + \cdots \\ &\cdots + hn_k J(K; 0, \cdots, 0, 1). \end{aligned} \tag{7}$$

Assuming (2) is true with $p > 0$, we can replace $J(K; 1, 0, \cdots, 0)$ and subsequent terms in (7) by the appropriate multiple of $W_1(K)$, retaining the inequality. After doing this, using $n = n_1 + \cdots + n_k$, and observing that $nW_1(K) = S(K) =$ the surface area of K, we obtain

$$J(K + hB; 0, \cdots, 0) \geqq J(K; 0, \cdots, 0) + h \frac{\omega_{n_1} \cdots \omega_{n_k}}{\omega_n} S(K). \tag{8}$$

We now use (8) to obtain an integral inequality. Let r be the radius of the largest ball contained in K, and for $0 \leqq \lambda \leqq r$, let K_λ be the inner parallel set of K at distance $r - \lambda$. That is, K_λ is the set of points of K whose distance from the boundary is not less than $r - \lambda$. As shown in [2, p. 147], each K_λ is convex, K_0 the kernel of K has dimension $\leqq n - 1$, and, for $0 \leqq \lambda \leqq \lambda + h \leqq r$, we have the important relation

$$K_{\lambda + h} \supset K_\lambda + hB. \tag{9}$$

The obvious monotonicity of J then yields

$$J(K_{\lambda + h}; 0, \cdots, 0) \geqq J(K_\lambda + hB; 0, \cdots, 0). \tag{10}$$

Using (8) we obtain then,

$$J(K_{\lambda + h}; 0, \cdots, 0) - J(K_\lambda; 0, \cdots, 0) \geqq h \frac{\omega_{n_1} \cdots \omega_{n_k}}{\omega_n} S(K_\lambda). \tag{11}$$

Let $0 \leqq \lambda_1 < \lambda_2 < \cdots < \lambda_m = r$ be any subdivision of the closed interval $[0, r]$. For each $i = 0, 1, \cdots, m - 1$, set $\lambda = \lambda_i$ and $h = \lambda_{i+1} - \lambda_i$ in (11) and sum the resulting inequalities to obtain

$$(12)\qquad J(K;0,\cdots,0)-J(K_0;0,\cdots,0)\geqq\frac{\omega_{n_1}\cdots\omega_{n_k}}{\omega_n}\sum_{i=0}^{m-1}S(K_{\lambda_i})(\lambda_{i+1}-\lambda_i).$$

Note that we have used the fact that $K_r = K$. The sum on the right-hand side of (12) is a Riemann sum for the integral with respect to λ of $S(K_\lambda)$ over the interval $[0, r]$. Since that integral is precisely $V(K)$ (see [2, p. 207]), we obtain from (12)

$$(13)\qquad J(K;0,\cdots,0)-J(K_0;0,\cdots,0)\geqq\frac{\omega_{n_1}\cdots\omega_{n_k}}{\omega_n}V(K),$$

which implies (2) with $p = 0$. This completes the proof.

It is of interest to note that if we do not drop terms of higher order in h, in (6) and (7), we obtain a formula of Steiner's type for $J(K + hB;\ \cdots)$.

The next lemma enables us to establish (2) for a given value of n and $p > 0$, provided we know (2) is valid for all smaller values of n and $p \geqq 0$.

LEMMA 2. *Let $n \geqq 2$ be given, and suppose* (2) *holds with $p \geqq 0$ and all values of n smaller than this given value. Then it follows that* (2) *holds for the given value of n and $p > 0$.*

PROOF. Let dE_q denote the normalized integralgeometric density for oriented q-dimensional subspaces of E_n. Then we have the following projection formula of Cauchy's type:

$$(14)\qquad \int \tilde{W}_r(K\,|\,E_{n-q})dE_{n-q}=\frac{\omega_{n-q}}{\omega_n}W_{r+q}(K).$$

This formula, valid for $0 \leqq r \leqq n-q \leqq n-1$, is equivalent to that found in [2, p. 232], except that here dE_{n-q} is the density for *normalized* Haar measure on oriented $(n-q)$-subspaces, and (14) still differs by a factor of 2 from the corresponding formula in [2], even after normalization, since unoriented subspaces are used there. Suppose $g \in SO(n)$ sends the standard basis to $u_1, \cdots, u_n$. Then $u_1, \cdots, u_{n-q}$ span an oriented $(n-q)$-subspace E_{n-q}, and $u_{n-q+1}, \cdots, u_n$ span the orthogonal q-subspace E_q. We can factor the density dg as follows:

$$(15)\qquad dg = dg_{n-q}dg_q dE_{n-q},$$

where dg_q is the normalized density for proper motions of E_q onto itself and dg_{n-q} the normalized density for proper motions of E_{n-q}. This factorization is established in a manner analogous to that in [2, p. 227], or directly by observing that the Grassmann manifold of oriented q-subspaces of E_n is the quotient space $SO(n)/SO(q) \times SO(n-q)$. This enables us to write a projection formula like (14) but involving integration over $SO(n)$. With $1 \leqq q \leqq n-1$, and $0 \leqq r \leqq n-q$, we have

$$\int \tilde{W}_r(K \mid E_{n-q})dg = \int \tilde{W}_r(K \mid E_{n-q})dg_{n-q}dg_q dE_{n-q}. \tag{16}$$

Now observe that with E_{n-q} fixed, the integrals with respect to dg_{n-q} and dg_q both contribute only a factor of 1. Hence we have, using (14)

$$\int \tilde{W}_r(K \mid E_{n-q})dg = \frac{\omega_{n-q}}{\omega_n} W_{r+q}(K). \tag{17}$$

Let us assume $p_1 > 0$. Letting dg_{n_1} be the density for motions in $E[n_1]$, dg_{n-n_1} the density for motions in the orthogonal subspace E_{n-n_1}, we may write, from (1),

$$\begin{aligned} J &= \int \tilde{W}_{p_1}\tilde{W}_{p_2}\cdots \tilde{W}_{p_k} dg_{n_1} dg_{n-n_1} dE_{n-n_1} \\ &= \int \left\{ \int \tilde{W}_{p_1}(K \mid E[n_1]) dg_{n_1} \right\} \tilde{W}_{p_2} \cdots \tilde{W}_{p_k} dg_{n-n_1} dE_{n-n_1} \end{aligned} \tag{18}$$

where for brevity we have omitted the parameters in some of the functions involved. Observe that the integral in the curly brackets in (18) is equal to $\tilde{W}_{p_1}(K \mid E[n_1])$ since the integrand is constant with respect to the integration. Now let $E_{n_1-p_1}$ be the subspace spanned by $u_{p_1+1}, u_{p_1+2}, \cdots, u_{n_1}$. Let us apply (17) to the space $E[n_1]$, setting $n = n_1$, $dg = dg_{n_1}$, $r = 0$, and $q = p_1$ (here we use $p_1 > 0$). We obtain then

$$\int \tilde{W}_0(K \mid E_{n_1-p_1})dg_{n_1} = \frac{\omega_{n_1-p_1}}{\omega_{n_1}} \tilde{W}_{p_1}(K \mid E[n_1]). \tag{19}$$

We used the fact that $K \mid E_{n_1-p_1}$ can be obtained as the projection of $K \mid E[n_1]$ into $E_{n_1-p_1}$. Now substituting $\tilde{W}_{p_1}(K \mid E[n_1])$ for the integral in curly brackets in (18), then substituting in turn for $\tilde{W}_{p_1}(K \mid E[n_1])$ from (19), and using $dg = dg_{n_1}dg_{n-n_1}dE_{n-n_1}$, we obtain

$$J = \frac{\omega_{n_1}}{\omega_{n_1-p_1}} \int \tilde{W}_0(K \mid E_{n_1-p_1})\tilde{W}_{p_2}\cdots \tilde{W}_{p_k}dg. \tag{20}$$

Now we factor dg in a different way. Let dg_{p_1} be the density for motions in the subspace spanned by $u_1, \cdots, u_{p_1}$, and dg_{n-p_1} the density for motions in the orthogonal subspace E_{n-p_1}. Then we have from (20)

$$J = \frac{\omega_{n1}}{\omega_{n_1-p_1}} \int \tilde{W}_0(K|\,E_{n_1-p_1})\tilde{W}_{p_2}\cdots \tilde{W}_{p_k} dg_{n-p_1} dg_{p_1} dE_{n-p_1} \tag{21}$$

$$= \frac{\omega_{n1}}{\omega_{n_1-p_1}} \int \tilde{J}(K\,|\,E_{n-p_1};\, n_1 - p_1, n_2, \cdots, n_k;\, 0, p_2, \cdots, p_k) dg_{p_1} dE_{n-p_1},$$

where $\tilde{J}$ has the obvious meaning of J restricted to E_{n-p_1}. Since by our hypothesis, (2) is true with n replaced by $n - p_1$, we have

$$\tilde{J}(K\,|\,E_{n-p_1};\, n_1 - p_1, n_2, \cdots, n_k;\, 0, p_2, \cdots, p_k) \tag{22}$$

$$\geqq \frac{\omega_{n_1-p_1}\omega_{n2}\cdots\omega_{nk}}{\omega_{n-p_1}} \tilde{W}_{p-p_1}(K\,|\,E_{n-p_1}).$$

Using this in (21), noting that the integral with respect to dg_{p_1} contributes only a factor of 1, and applying (14) again, we have

$$J \geqq \frac{\omega_{n1}\cdots\omega_{nk}}{\omega_{n-p_1}} \int \tilde{W}_{p-p_1}(K|\,E_{n-p_1}) dE_{n-p_1} \tag{23}$$

$$= \frac{\omega_{n1}\cdots\omega_{nk}}{\omega_{n-p_1}} \frac{\omega_{n-p_1}}{\omega_n} W_p(K),$$

proving that (2) holds under the given hypotheses. Since clearly a similar argument could have been used assuming any $p_i > 0$, this completes the proof.

3. Proof of the main theorem

We shall first establish that (2) is true and later consider when equality can hold. Consider first the case $n = 2$, $n_1 = n_2 = 1$. If $p_1 = p_2 = 1$, then equality holds in (2), both sides being equal to 4. If $p_1 = 1$ and $p_2 = 0$, or vice-versa, we again have equality in (2) since we then simply have the fact that the perimeter of a convex curve is π times its mean width. Lemma 1 now shows that (2) holds with $p_1 = p_2 = 0$. Now suppose $m > 2$ is given and we know that (2) holds for $n < m$ and $p \geqq 0$. Then Lemma 2 implies that (2) holds for $n = m$ and $p > 0$. It then follows from Lemma 1 that (2) holds for $n = m$ and $p \geqq 0$. It follows by induction that (2) is true in all dimensions.

We shall also use induction to establish when equality can hold in (2). In case $n = 2, n_1 = n_2 = 1$, we have seen that we have equality for all K unless $p_1 = p_2 = 0$. In the latter case, as shown in [1], and [4], equality holds only for a circular disk. Now suppose $m > 2$ is given. Suppose that it has been proved that in $n < m$, and $p_i \neq n_i$ for at least two different values of i, then equality can hold in (2) only for a ball. Let K be a convex body in E_m for which equality holds in (2) with $p_i \neq m_i$ for at least two different values of i (where we now write $m = m_1 + \cdots + m_k$

rather than $n = n_1 + \cdots + n_k$), and suppose $p_1 > 0$ (in a moment we shall consider the possibility $p = 0$). An examination of the proof of Lemma 2 shows that equality would have to hold in (22) (with n's replaced by m's). Since some pair of $0, p_2, p_3, \cdots, p_k$ must be respectively less than the corresponding pair of $m_1 - p_1$, $m_2, \cdots, m_k$, our induction hypothesis implies that $K \mid E_{m-p_1}$ is a ball for all $(m - p_1)$-subspaces E_{m-p_1}. Note that $m - p_1 \geqq 2$, since otherwise $p_1 \geqq m - 1$ so $m_1 = m - 1$, $m_2 = 1$, and we would not have $p_i \neq m_i$ for at least two different values of i, contrary to hypothesis. Thus the subspaces E_{m-p_1} have dimension at least 2, and since the orthogonal projection of K on each such subspace is a ball, it is easy to show K itself must be a ball. Of course we could have obtained the same result assuming any $p_i > 0$, rather than $p_1 > 0$. Now suppose K is a convex body in E_m for which equality holds in (2), with $p = 0$. Then an examination of the proof of Lemma 1 shows that we must have equality in (2) whenever we set $p_i = 1$ and $p_j = 0, j \neq i, i = 1, 2, \cdots, k$. Our previous argument then shows K must be a ball. By induction it now follows that if $p_i \neq n_i$, for at least two different values of i, then equality can hold in (2) only if K is a ball.

If $p_i = n_i$ for all i, then both sides of (2) are equal to the product $\omega_{n_1} \cdots \omega_{n_k}$, so equality holds for all K. If $p_i = n_i$ with only one exception, then (2), with an equality sign, is equivalent to the projection formula (17), with appropriate parameters, so one again has equality for all K. This completes the proof of Theorem 1.

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